Dark energy as space-time curvature induced by quantum vacuum fluctuations

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Abstract

It is shown that quantum vacuum fluctuations give rise to a curvature of space-time equivalent to a cosmological constant, that is a homogeneous energy density ρ and pressure p fulfilling -$p = \rho > 0$. The fact that the fluctuations produce curvature, even if the vacuum expectation of the energy vanishes, is a consequence of the non-linear character of the Einstein equation. A calculation is made, involving plausible hypotheses within quantized gravity, which establishes a relation between the two-point correlation of the vacuum fluctuations and the space-time curvature. Arguments are given which suggest that the density ρ might be of order the "dark energy" density currently assumed to explain the observed accelerated expansion of the universe.

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1 Introduction

Recent astronomical observations, in particular the study of type Ia supernovae, anisotropies in the cosmic background radiation and matter power spectra inferred from large galaxy surveys, have improved our knowledge of the universe giving rise to a precision cosmology. The new data are compatible with the universe having a Friedmann - Robertson - Walker metric with

flat spatial slices[1] of the form

$$ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2 d\Omega\right). \qquad (1)$$

The time-dependent parameter $a(t)$ is related, at present time t_0, to the measurable Hubble constant, H_0, and decceleration parameter, q_0, via

$$\left[\frac{\dot{a}}{a}\right]_{t_0} = H_0, \quad \left[\frac{\ddot{a}}{a}\right]_{t_0} = -H_0^2 q_0. \qquad (2)$$

The observations also provide information about the evolution of the parameter $a(t)$ and the distribution of matter in the past.

The available knowledge is summarized in the ΛCDM model. In it baryonic matter density, ρ_B, represents about 4.6% of the matter content of the universe while two hypothetical ingredients named cold dark matter (CDM) and dark energy (DE) contribute with densities $\rho_{DM} \sim 23\%$, and $\rho_{DE} \sim 73\%$ respectively[2]. The said densities are related to the metric eq.(1) via the Friedmann equations, derived from general relativity, giving the following relations[3]

$$\begin{aligned}
\left[\frac{\dot{a}}{a}\right]^2 &= \frac{8\pi G}{3}\left(\rho_B(t) + \rho_{DM}(t) + \rho_{DE}\right), \\
\frac{\ddot{a}}{a} &= \frac{8\pi G}{3}\left(\frac{1}{2}\left[\rho_B(t) + \rho_{DM}(t)\right] - \rho_{DE}\right),
\end{aligned} \qquad (3)$$

where I neglect small effects of radiation and matter pressure. The baryonic density ρ_B is well known from the measured abundances of light chemical elements, which allows calculating ρ_{DE} and ρ_{DM} from the empirical quantities H_0 and q_0 via comparison of eqs.(3) and (2). The values obtained by this method agree with data from other observations. For instance cold dark matter, in an amount compatible with ρ_{DM}, is needed in order to explain the observed (almost flat) rotation curves in galaxies. However the nature of dark matter and dark energy remain open problems.

Usually dark matter is assumed to derive from exotic particles, not yet discovered, and dark energy is identified with a cosmological constant, Λ. In any case the ΛCDM model rest upon the assumption that general relativity (GR) is indeed the correct theory of gravity. However it is conceivable that both cosmic speed up and dark matter represent signals of a breakdown of GR. For instance we might consider the possibility that the Hilbert - Einstein

Lagrangian, linear in the Ricci scalar R, should be generalized to become a function $f(R)$. This is the underlying philosophy of what is referred to as $f(R)$ gravity theory[4]. Indeed the cosmological constant corresponds to a particular choice of $f(R)$ where a constant Λ is added to the Ricci scalar R, although this does not give any hint about the value of ρ_{DE}. The theory of $f(R)$ gravity provides sufficient freedom to accomodate also dark matter. For instance it allows good fits to the rotation curves in galaxies[5], which therefore might be explained as a curvature effect. Actually $f(R)$ gravity may be further generalized by including other scalars, like $R_{\mu\nu}R^{\mu\nu}$, in the Hilbert - Einstein Lagrangian[6]. In this paper I shall not deal with dark matter, but only with dark energy, so that I will take the density ρ_{DM} as an empirical datum, without any discussion about its possible nature.

Many proposals have been made for the origin of dark energy (for a review see Copeland[7]). As said above the most popular one is to identify it with a cosmological constant or, what is equivalent in practice, to assume that it derives from the quantum vacuum. Indeed the term ρ_{DE} in eqs.(3) might be interpreted as coming from a vacuum energy whose pressure fulfils $p_{DE} = -\rho_{DE}$, an equality appropriate for the vacuum (in Minkowski space, or when the space-time curvature is small) because it is invariant under Lorentz transformations. A problem appears however when one attempts to estimate the value of ρ_{DE}. For instance if the dark energy is due to the interplay between quantum mechanics and gravity, it may seem that it should be either strictly zero or of order Planck´s density, that is

$$\rho_{DE} \sim \frac{c^5}{G^2\hbar} \simeq 10^{97} \text{ kg/m}^3. \tag{4}$$

In sharp contrast the observations lead to

$$\rho_{DE} \simeq 10^{-26} \text{ kg/m}^3, \tag{5}$$

a value departing from eq.(4) by about 123 orders of magnitude. This strong disagreement gives rise to the "cosmological constant problem"[8].

In this paper I will explain why the value of ρ_{DE} can be much smaller than eq.(4) even if the cosmological constant really derives from the quantum vacuum and I shall do that without departing from standard general relativity. In fact I shall prove, modulo a few plausible assumptions within quantum gravity, that vacuum fluctuations give rise to a curvature of space-time fully equivalent to the one produced by a cosmological constant, even

if the vacuum expectation of the density of quantum fields is strictly zero. The argument is as follows.

For a small enough region of the universe around us, but large in comparison with typical distances between galaxies, the space-time metric given by eqs.(1) may be rewritten, near present time, using new coordinates as follows[3]

$$ds^2 \simeq g_{rr}dr^2 + r^2 d\Omega^2 - g_{tt}dt^2, g_{rr} = \left[1 + \left[\frac{\dot{a}}{a}\right]_0^2 r^2\right], g_{tt} = \left[1 + \left[\frac{\ddot{a}}{a}\right]_0 r^2\right] \quad (6)$$

where terms of order $O\left(r^4\right)$ have been neglected and it is assumed that the (slow) change of the metric coefficients with time may be ignored. This metric is Minkowskian for small r, which makes the calculations more simple than using eq.(1). In this paper I will calculate the coefficients g_{rr} and g_{tt} of eq.(6) as coming from the combined action of cold matter, having homogeneous density $\rho_B + \rho_{DM}$ (at the large scale), plus the effect of the vacuum fluctuations. In my approach *the Friedmann eqs.*(3) *are not valid* because they were derived under the assumption that the space-time curvature, resulting in the metric eq.(6) , comes from a mixture of three fluids with total density $\rho_B + \rho_{DM} + \rho_{DE}$ and total pressure $-\rho_{DE}$. Here I will assume only two fluid with total density $\rho_B + \rho_{DM}$ and negligible pressure. The calculation leads to the following relations

$$\begin{aligned} g_{rr} &= 1 + \left[\frac{8\pi G}{3}\left(\rho_B(t) + \rho_{DM}(t)\right) + \Lambda_{fluct}\right] r^2 + O\left(r^4\right), \\ g_{tt} &= 1 + \left[\frac{8\pi G}{3}\left(\frac{1}{2}\left[\rho_B(t) + \rho_{DM}(t)\right]\right) - \Lambda_{fluct}\right] r^2 + O\left(r^4\right), \end{aligned} \quad (7)$$

where Λ_{fluct} is a constant parameter with dimentions of inverse length squared. It is explicitly calculated from the two-point correlation of the vacuum fluctuations (see below). The net result is that the fluctuations produce the same effect on the space-time curvature as a cosmological constant. Thus eq.(7) may be written in a form similar to Friedman´s eqs.(3) provided we define a new quantity, ρ_{DE}, with dimensions of density as follows

$$\rho_{DE} \equiv \frac{3}{8\pi G}\Lambda_{fluct}. \quad (8)$$

But ρ_{DE} *is not any actual density, but a parameter taking into account the effect of the quantum vacuum fluctuations on space-time curvature.* My derivation does not provide a precise value of ρ_{DE} but it strongly suggests that it

is far smaller than eq.(4). The difficulty for getting the value is that the two-point correlation of the vacuum fluctuations is not known. If future calculations along this line provide a value of ρ_{DE} in agreement with eq.(5) then the universe speed up would be fully explained as due to the quantum vacuum fluctuations.

2 The two-point correlation function of vacuum fluctuations

The starting point of the work is an idea of Zeldovich[9], who proposed the relations

$$\rho_{DE}c^2 \sim G\frac{m^6c^4}{\hbar^4} = \frac{Gm^2}{\lambda} \times \frac{1}{\lambda^3}, \lambda \equiv \frac{\hbar}{mc} \tag{9}$$

where m is a typical mass of elementary particles, eq.(5) being obtained if the mass m is

$$m \sim 7.6 \times 10^{-29} \text{ kg},$$

that is about one third the pion mass. In my opinion the Zeldovich´s relation between cosmology and particle physics suggests that dark energy density eq.(5) does not correspond to the mean vacuum energy, which is likely zero, but to (small) gravity effects associated to the quantum vacuum fluctuations.

The second eq.(9) suggests that $\rho_{DE}c^2$ may have the magnitude of the gravitational energy of quantum vacuum fluctuations. This may be seen more explicitly using a semiclassical Newtonian theory of gravity, that is taking matter as quantized but the gravitational field as classical. Thus we may calculate the vacuum expectation of the gravitational energy associated to a sphere of radius R, that is

$$E = -G\int_{|\mathbf{r}_2|\leq R} d^3\mathbf{r}_1 \int_{|\mathbf{r}_2|\leq R} d^3\mathbf{r}_2 \frac{\frac{1}{2}\left\langle vac \left| \hat{\rho}\left(\mathbf{r}_1,t\right) \hat{\rho}\left(\mathbf{r}_2,t\right) + \hat{\rho}\left(\mathbf{r}_2,t\right) \hat{\rho}\left(\mathbf{r}_1,t\right) \right| vac \right\rangle}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{10}$$

This calculation rests upon the assumption that it is possible to define an energy density operator, $\hat{\rho}\left(\mathbf{r},t\right)$, of the quantum fields and that the vacuum expectation of that energy is zero but the expectation of the square is not zero. Actually if the expectation of $\hat{\rho}^2$ was also zero there would be no

quantum fluctuations at all. That is we must assume

$$\left\langle vac\left|\overset{\wedge}{\rho}\left(\mathbf{r},t\right)\right|vac\right\rangle =0,\ \left\langle vac\left|\overset{\wedge}{\rho}^{2}\right|vac\right\rangle \neq 0. \tag{11}$$

This being the case, by continuity we expect a two-point correlation function which may depend only on the distance $\left|\mathbf{r}_2-\mathbf{r}_1\right|$ for equal times in the non-relativistic approach leading to eq.(10). In a relativistic theory the correlation should depend on the interval, s, this being the only invariant in Minkowski space. The operators $\overset{\wedge}{\rho}\left(\mathbf{r}_1,t_1\right)$ and $\overset{\wedge}{\rho}\left(\mathbf{r}_2,t_2\right)$ may not commute and it is plausible to define the two-point correlation with the operators in symmetrical order, which guarantees that the correlation, $C(s)$, is real, without an imaginary part. That is

$$\begin{aligned} C\left(s\right) &= \frac{1}{2}\left\langle vac\left|\overset{\wedge}{\rho}\left(\mathbf{r}_1,t_1\right)\overset{\wedge}{\rho}\left(\mathbf{r}_2,t_2\right)+\overset{\wedge}{\rho}\left(\mathbf{r}_2,t_2\right)\overset{\wedge}{\rho}\left(\mathbf{r}_1,t_1\right)\right|vac\right\rangle, \\ s^2 &= \left(\mathbf{r}_1-\mathbf{r}_1\right)^2-\left(t_1-t_2\right)^2, \end{aligned} \tag{12}$$

with units $c=1$ which I shall use from now on. With this definition the correlation might depend on whether the interval s is space-like or time-like, in the latter case s being imaginary.

Taking eq.(12)into account and considering equal times, $t_1=t_2$, eq.(10) leads to

$$\begin{aligned} E &= -G\int_{\left|\mathbf{r}_2\right|\leq R}d^3\mathbf{r}_1\int_{\left|\mathbf{r}_2\right|\leq R}d^3\mathbf{r}_2\frac{C\left(\left|\mathbf{r}_1-\mathbf{r}_2\right|\right)}{\left|\mathbf{r}_1-\mathbf{r}_2\right|} \\ &\simeq -G\int_{\left|\mathbf{r}_2\right|\leq R}d^3\mathbf{r}_1\int_0^{\infty}4\pi r^2\frac{C\left(r\right)}{r}dr=-4\pi GV\int_0^{\infty}C\left(r\right)rdr, \end{aligned} \tag{13}$$

where r stands for $\left|\mathbf{r}_1-\mathbf{r}_2\right|$ and the integral in r has been extended to ∞ because we assume that the radius R is much larger than the range of the correlation function $C\left(r\right)$. The result shows that, in Newtonian gravity, the existence of fluctuations necessarily implies a gravitational energy associated to them, with density

$$\rho_{grav}=-4\pi G\int_0^{\infty}C\left(r\right)rdr. \tag{14}$$

The gravitational energy appears in spite of the fact that the vacuum expectation of the matter density operator vanishes everywhere, see eq.(11).

The fact that semiclassical Newtonian theory predicts that vacuum fluctuations give rise to a gravitational energy, even if the mean energy density of the vacuum is strictly zero, shows that in semiclassical general relativity the vacuum fluctuations will produce space-time curvature. Indeed in general relativity the concepts of gravitational force and gravitational energy lose their meaning and the relevant concept is the curvature of space-time.

The aim of this paper is to present a calculation using general relativity. Thus the goal of our calculation will be to find the curvature of space-time induced by the quantum vacuum fluctuations. The calculation develops an idea put forward elsewhere[11]. Before presenting the calculation I shall discuss further the subject of the two-point correlations of vacuum energy densities. This is necessary because quantum vacuum fluctuations might be seen as artifacts of the quantum formalism, without real physical implications, in view that in most cases they may be eliminated by using normal ordering of the creation and annihilation operators.

The two-point function $C(s)$ might be calculated in flat (Minkowski) space-time from the properties of quantum fields in vacuum, but making a calculation which involves all known fields would be a formidable task. Nevertheless the calculation is straightforward in principle, as shown by the derivation which follows of the contribution due to the free electromagnetic field, which I will perform for illustrative purposes. In quantum theory the vacuum expectation of the energy of any unexcited field is assumed to be zero and this assumption is stated formally by using the normal ordering of the operators. For instance in the electromagnetic field we have

$$\langle vac \left| \widehat{\rho}\left(\mathbf{r},t\right)\right| vac\rangle = 0,\ \widehat{\rho}\left(\mathbf{r},t\right) \equiv\ : \frac{\widehat{\mathbf{E}}\left(\mathbf{r},t\right)^2+\widehat{\mathbf{H}}\left(\mathbf{r},t\right)^2}{8\pi} :\ , \tag{15}$$

where normal ordering implies that the energy density operator $\widehat{\rho}$ contains only products of creation, $\widehat{a}^{+}$, and annihilation, $\widehat{a}$, operators of photons of the type $\widehat{a}\widehat{a}, \widehat{a}^{+}\widehat{a}^{+}$ y $\widehat{a}^{+}\widehat{a}$, all of which give a nil vacuum expectation. Therefore the vacuum expectation of the energy density is zero as assumed. However the two-point correlation is not zero because the operator

$$\widehat{C}\left(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2\right) \equiv (: \frac{\widehat{\mathbf{E}}\left(\mathbf{r}_1,t_1\right)^2+\widehat{\mathbf{H}}\left(\mathbf{r}_1,t_1\right)^2}{8\pi} :)(: \frac{\widehat{\mathbf{E}}\left(\mathbf{r}_2,t_2\right)^2+\widehat{\mathbf{H}}\left(\mathbf{r}_2,t_2\right)^2}{8\pi} :), \tag{16}$$

contains terms of the type $\widehat{a}\widehat{a}\widehat{a}^{+}\widehat{a}^{+}$ whose vacuum expectation is finite.

The two-point correlation function of the free electromagnetic field will be just the vacuum expectation of eq.(16). The calculation is straightforward

using the plane-waves expansions

$$\begin{aligned}\widehat{E}(\mathbf{r},t) &= \sum_{\mathbf{k}\boldsymbol{\varepsilon}}\left(\frac{\hbar\omega}{2V}\right)^{1/2}\left[\widehat{a}_{\mathbf{k}\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}(\mathbf{k})\exp(i\mathbf{k}.\mathbf{r}-i\omega t)+\widehat{a}_{\mathbf{k}\boldsymbol{\varepsilon}}^{+}\boldsymbol{\varepsilon}(\mathbf{k})\exp(-i\mathbf{k}.\mathbf{r}+i\omega t)\right], \\ \widehat{H}(\mathbf{r},t) &= \sum_{\mathbf{k}\boldsymbol{\varepsilon}}\left(\frac{\hbar}{2V\omega}\right)^{1/2}[\widehat{a}_{\mathbf{k}\boldsymbol{\varepsilon}}(i\mathbf{k}\times\boldsymbol{\varepsilon}(\mathbf{k}))\exp(i\mathbf{k}.\mathbf{r}-i\omega t) \\ &\quad -\widehat{a}_{\mathbf{k}\boldsymbol{\varepsilon}}^{+}(i\mathbf{k}\times\boldsymbol{\varepsilon}(\mathbf{k}))\exp(-i\mathbf{k}.\mathbf{r}+i\omega t)], \end{aligned} \tag{17}$$

with standard notation (V is a normalization volume and $\omega \equiv |\mathbf{k}|$). The details may be seen in the Appendix and the result is

$$C(s)=\frac{2\hbar^2}{\pi^4 s^8}, s^2 \equiv r^2-t^2. \tag{18}$$

The two-point correlation depends on distance and time interval only via r^2-t^2 as it should. Furthermore it depends on $|r^2-t^2|$, so making no distinction between time-like and space-like intervals.

The correlation $C(s)$, eq.(18), decreases rapidly at large $|r^2-t^2|$ but has a strong divergence when $r^2 \to t^2$. It is plausible to assume that quantum fields other than the electromagnetic one will give rise to counterterms which eliminate the divergence. In particular the sums in $\mathbf{k}$ which appear in eqs.(17) have been extended to very large values of $|\mathbf{k}|$, but this is physically absurd. Indeed the plane waves expansions eqs.(17) correspond to assuming an energy $\frac{1}{2}\hbar\omega$ per normal mode of the radiation but, for values of $\hbar\omega$ larger than two electron masses, the electromagnetic field may produce electron-positron pairs and the study of the radiation field alone makes no sense. This may be also stated saying that at very high frequencies the electromagnetic vacuum is polarized, so that the denominators 8π in eq.(15) should be replaced by larger quantities. For a rigorous treatment we should study the electromagnetic field interacting with the electron-positron field, but we should also include all other charged particles, and all particles interacting with these via strong of weak forces. In summary, a consistent calculation of the correlation function should involve *all* quantum fields (excluding gravity.) In the absence of that calculation I shall assume that the tail of the correlation function $C(s)$ is given by eq.(18), the photon being the only known massless particle which may exist freely, but that at small values of s the function $C(s)$ remains finite. If this is the case, it implies that the contributions of particles with different masses are not additive, but may cancel each other to some extent.

Another mechanism able to remove the strong divergence of eq.(18) will be discussed at the end of this section. A simple form to take into account these possibilities is to introduce a cut-off, substituting the following for eq.(18)

$$C(s)=\frac{2\hbar^{2}}{\pi^{4}\left(s^{2}+\lambda^{2}\right)^{4}},\lambda=\frac{\hbar}{m},$$

where m is an unknown mass. Putting this into eq.(13) we get

$$\rho_{grav}=-\frac{4}{3\pi^{3}}\frac{Gm^{6}}{\hbar^{4}},$$

close to Zeldovich´s proposal eq.(9) . It is not easy to estimate the value of the cut-off mass m without a detailed knowledge of the correlation function $C(s)$, eq.(12), but we may guess that m is of order the masses of the fundamental particles of the "standard model" rather than of order Planck´s mass. Thus a model resting upon Zeldovich´s idea predicts a density ρ_{DE} much closer, if not identical, to eq.(5) than to eq.(4) .

The dependence of the two-point correlation on the interval s gives rise to a paradox which reflects the counterintuitive feature that the correlation does not decrease with distance. It may be seen as a straightforward prediction of relativistic theory, but I shall show in the following that it is really counterintuitive. The paradox may be stated as follows. Let us assume, for the sake of simplicity, that the density vacuum fluctuation in a point possesses discrete values, say ρ_1 with probability P_1, ρ_2 with probability P_2, etc. Then the mean square fluctuation will be (compare with eq.(12))

$$C\left(0\right)=\sum_{j}P_{j}\rho_{j}^{2}. \tag{19}$$

Now we consider two different points, the correlation of fluctuations being

$$C\left(s\right)=\sum_{j}\sum_{k}p_{jk}\rho_{j}\rho_{k}, \tag{20}$$

where p_{jk} is the probability that the vacuum fluctuation in the first point is ρ_j and the fluctuation in the second point it is ρ_k. If the two said points are separated by a light-like interval, then $s=0$ so that eqs.(19) and (20) lead to

$$C\left(0\right)=\sum_{j}\sum_{k}p_{jk}\rho_{j}\rho_{k}=\sum_{j}P_{j}\rho_{j}^{2}=\sum_{j}\sum_{k}p_{jk}\rho_{j}^{2},$$

where the latter equality follows from well known properties of the probabilities. Hence it is trivial to derive the equality

$$\sum_j \sum_k p_{jk} \left(\rho_j^2 + \rho_k^2 - 2\rho_j \rho_k\right) = \sum_j \sum_k p_{jk} \left(\rho_j - \rho_k\right)^2 = 0.$$

For (positive) probabilities this equality can be true only if $p_{jk} = 0$ for any $j \neq k$. This means that the fluctuations are strictly correlated in the whole light cone of every point. Furthermore, for two arbitrary points, S_1 and S_2, it is always possible to find another point S which is light-like separated from each one. In fact, all points in the intersection of the light cones of S_1 and S_2 do the job. As a consequence for all pairs of points the probabilities p_{jk} are zero for any $j \neq k$. which implies that vacuum fluctuations are *strictly correlated at all points in space-time*!. This conclusion, asides from being highly counterintuitive, contradicts known facts about quantum fluctuations. A possible solution to the paradox is that correlations between events in different points of space cannot be written, as in eq.(20) , using joint probabilities, a well known fact in quantum mechanics (for instance, it is crucial in the proof of Bell´s theorem[10].) There is another solution (which does not exclude the former), namely that Minkowski space is not well defined in quantized general relativity. In fact, in quantized gravity the metric should be quantized, meaning that the metric coefficients are operators (see eq.(27) below). Thus neither the distance nor the time interval between events are well defined. In other words, given two events of coordinates $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$ there is a quantum uncertainty about the relativistic interval existing between them. It is possible to state with confidence that two events are spatially separated if $|\mathbf{r}_1 - \mathbf{r}_2| >> |t_1 - t_2|$ or temporally separated if $|\mathbf{r}_1 - \mathbf{r}_2| << |t_1 - t_2|$, but it is never possible to state that they are light-like separated. I point out that this fact already removes the divergence of the two-point correlation function, shown e. g. in eq.(18) .

3 Space-time curvature due to quantum vacuum fluctuations

Working within quantized gravity the space-time structure is determined by the quantum state, $\mid \Phi\rangle$, of the universe and the matter stress-energy tensor operator, $\widehat{T}_{\mu\nu}(x)$, of the quantum fields at every space-time point, x. Here

x stands for the 4 coordinates in an appropriate reference frame, that is

$$x \equiv \{x_1, x_2, x_3, x_4\}\,, \tag{21}$$

The study of the quantum fields in curved space-times and the gravitational back reaction of the fields is a difficult subject[12]. In particular the curvature may give rise to a modification of the vacuum stress-energy[6]. However for our purposes the metric is so close to Minkowskian that we may treat the quantum fields as if they existed in flat space-time, although we want to calculate the (small) curvature induced by the vacuum fluctuations of the fields.

Our approach rests upon the existence of two quite different scales in the problem, namely a *cosmic* scale (with typical distances of megaparsecs) and the *atomic* scale of the correlations between vacuum fluctuations (which involves distances smaller than, say, nanometers). In the latter scale quantization is essential, but in the former we may treat everything as classical, as is explained in the following. For any two quantum observables, $\widehat{a}\,(x)$ and $\widehat{b}\,(y)$, at the space-time points x and y respectively, we may define the correlation

$$C_{ab}\,(x,y) \equiv \langle\Phi \mid \widehat{a}\,(x)\,\widehat{b}\,(y) \mid \Phi\rangle - \langle\Phi \mid \widehat{a}\,(x) \mid \Phi\rangle\langle\Phi \mid \widehat{b}\,(y) \mid \Phi\rangle. \tag{22}$$

Now it is plausible to assume that the correlation may be relevant at the atomic scale but goes to zero when the distance increases toward a macroscopic scale. As a consequence, in the cosmic scale we may treat the expectations of quantum observables as classical variables, and the expectations of products of observables as products of the corresponding classical variables. For instance

$$\langle\Phi \mid \widehat{a}\,(x)\,\widehat{b}\,(y) \mid \Phi\rangle \simeq a(x)b(y),\;\; a(x) \equiv \langle\Phi \mid \widehat{a}\,(x) \mid \Phi\rangle,\;\; b(y) \equiv \langle\Phi \mid \widehat{b}\,(y) \mid \Phi\rangle. \tag{23}$$

In summary, we may ignore quantization when working with problems at any macroscopic scale provided we use as classical variables the expectations of the corresponding quantum observables. In sharp contrast, at the atomic scale we should work within quantized gravity. This happens in particular when $x = y$, that is

$$\langle\Phi \mid \widehat{a}\,(x)\,\widehat{b}\,(x) \mid \Phi\rangle \neq \langle\Phi \mid \widehat{a}\,(x) \mid \Phi\rangle \times \langle\Phi \mid \widehat{b}\,(x) \mid \Phi\rangle.$$

The main hypothesis of this paper is that the expectation of the stress-energy tensor operator of the quantum fields at any point gives the matter

(baryonic or dark) stress-energy, without any additional contribution of the vacuum. With reference to eqs.(3) and (6), this means that

$$\langle \Phi \mid \widehat{T}_0^0 \mid \Phi \rangle = \rho_{mat},\ \langle \Phi \mid \widehat{T}_\mu^\nu \mid \Phi \rangle \simeq 0 \text{ for } \mu\nu \neq 00. \tag{24}$$

This suggests defining a vacuum stress-energy tensor operator as

$$\widehat{T}_{\mu\nu}^{vac} \equiv \widehat{T}_{\mu\nu} - \langle \Phi \mid \widehat{T}_{\mu\nu} \mid \Phi \rangle \widehat{I} \equiv \widehat{T}_{\mu\nu} - T_{\mu\nu}^{mat} \widehat{I} \tag{25}$$

where $\widehat{I}$ is the identity operator. The existence of vacuum fluctuations means that, although the expectation of $\widehat{T}_{\mu\nu}^{vac}$ is zero by definition, there are correlated vacuum fluctuations, that is

$$\left\langle \Phi \left| \widehat{T}_{\mu\nu}^{vac}(x)\, \widehat{T}_{\lambda\sigma}^{vac}(y) \right| \Phi \right\rangle \neq 0 \text{ in general.} \tag{26}$$

In order to proceed with the calculation I shall start with the quantum metric

$$d\widehat{s}^2 = \widehat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{27}$$

using polar coordinates

$$x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \phi, \tag{28}$$

and I shall write the (quantum operators) coefficients of the metric in the form

$$\begin{aligned} \widehat{g}_{00} &= -1 + \widehat{h}_{00},\ \widehat{g}_{11} = 1 + \widehat{h}_{11},\ \widehat{g}_{22} = r^2 \left(1 + \widehat{h}_{22}\right), \\ \widehat{g}_{33} &= r^2 \sin^2\theta \left(1 + \widehat{h}_{33}\right),\ \widehat{g}_{\mu\nu} = \widehat{h}_{\mu\nu} \text{ for } \mu \neq \nu, \end{aligned} \tag{29}$$

(multiplication of every term times the unit operator is implicit). Here $\widehat{h}_{\mu\nu}$ is a (small in some sense) correction to a Minkowski metric. If we want that the vacuum expectation of eq.(27) agrees with eqs.(3) to (8) we should have, to order $O\left(r^2\right)$,

$$\begin{aligned} \left\langle \widehat{h}_{\mu\nu} \right\rangle &= 0 \text{ except } \left\langle \widehat{h}_{00} \right\rangle = \frac{8\pi G}{3} \left(\rho_{DE} + \rho_{mat}\right) r^2, \\ \left\langle \widehat{h}_{11} \right\rangle &= \frac{8\pi G}{3} \left(\rho_{DE} - \frac{1}{2}\rho_{mat}\right) r^2, \end{aligned} \tag{30}$$

where $\left\langle \widehat{h}_{\mu\nu} \right\rangle$ stands for $\left\langle \Phi \left| \widehat{h}_{\mu\nu} \right| \Phi \right\rangle$.

The proof will consist of the following steps:

1. We should define an Einstein quantum tensor operator $\widehat{G}_{\mu\nu}$ in terms of the operators $\widehat{g}_{\mu\nu}$ (or what is equivalent, the operators $\widehat{h}_{\mu\nu}$).

2. Assuming that in quantized gravity the counterpart of Einstein equations reads

$$\widehat{G}_{\mu\nu} = \frac{8\pi G}{c^4}\widehat{T}_{\mu\nu}, \tag{31}$$

we should solve these (non-linear coupled partial differential) operator equations in order to get the quantum metric coefficients $\widehat{g}_{\mu\nu}$ in terms of integrals involving the stress-energy tensor operators $\widehat{T}_{\mu}^{\nu}(x)$ and products like $\widehat{T}_{\mu}^{\nu}(x)\,\widehat{T}_{\lambda}^{\sigma}(y)$, $\widehat{T}_{\mu}^{\nu}(x)\,\widehat{T}_{\lambda}^{\sigma}(y)\,\widehat{T}_{\rho}^{\tau}(z)$, etc.

3. Finally we should calculate the expectation of the metric coefficients $\widehat{g}_{\mu\nu}$ in terms of integrals involving the expectations

$$\left\langle \Phi \left| \widehat{T}_{\mu}^{\nu}(x) \right| \Phi \right\rangle, \left\langle \Phi \left| \widehat{T}_{\mu}^{\nu}(x)\,\widehat{T}_{\lambda}^{\sigma}(y) \right| \Phi \right\rangle, \left\langle \Phi \left| \widehat{T}_{\mu}^{\nu}(x)\,\widehat{T}_{\lambda}^{\sigma}(y)\,\widehat{T}_{\rho}^{\tau}(z) \right| \Phi \right\rangle, etc.$$

The expectation of the metric should reproduce eqs.(30).

A problem appears in the first step because there is not yet a quantum gravity theory specifying $\widehat{G}_{\mu\nu}$ in terms of $\widehat{g}_{\mu\nu}$, which would involve a quantum counterpart of Riemann´s theory. I will not solve the problem in general, but for the approximate expression of $\widehat{G}_{\mu\nu}$ containing only terms linear or quadratic in the (small) operators $\widehat{h}_{\mu\nu}$, I will make the plausible assumption that *the expression of $\widehat{G}_{\mu\nu}$, in terms of $\widehat{h}_{\mu\nu}$, and their derivatives with respect to the coordinates, is the same as the one for the corresponding classical quantities with the rule that the operators should appear in symmetrical order.* The latter assumption means that the operator corresponding to the classical product ab will be the quantum expression $\frac{1}{2}\left(\hat{a}\widehat{b}+\widehat{b}\hat{a}\right)$.

4 Quantum Einstein equation and its solution

In order to simplify de calculations I will introduce the approximation of retaining, in the expression of $\widehat{G}_{\mu\nu}$, only terms of zeroth and first order in $\widehat{h}_{\mu\nu}$, except for both $\widehat{h}_{00}$ and $\widehat{h}_{11}$, which will be mantained up to second order. With these approximations our calculation simplifies substantially by the following reasons. Firstly it may be realized that terms of zeroth order

will not contribute to $\widehat{G}_{\mu\nu}(x)$ because to zeroth order the metric eq.(29) is Minkowskian. In addition, the terms linear in $\widehat{h}_{\mu\nu}$ with $\mu\nu \neq 00$ and $\mu\nu \neq 11$ (and of zeroth order in both $\widehat{h}_{00}$ and $\widehat{h}_{11}$) will not contribute to the expectations $\left\langle \widehat{G}_{\mu\nu}(x) \right\rangle$ and $\left\langle \widehat{G}_{\mu\nu}(x) \widehat{G}_{\lambda\sigma}(y) \right\rangle$ when eqs.(30) are taken into account. Consequently we may ignore such terms from now on, which in practice is equivalent to putting $\widehat{h}_{\mu\nu} = 0$ whenever $\mu\nu \neq 00$ and $\mu\nu \neq 11$. This amounts to replacing the metric eq.(29) by

$$d\widehat{s}^2 = \exp\left(\widehat{\alpha}\right) dr^2 + (r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)\widehat{I} - \exp\left(\widehat{\beta}\right) dt^2, \tag{32}$$

where $\widehat{I}$ is the identity operators and I have introduced the new functions

$$\widehat{\alpha} \equiv \log\left(1 + \widehat{h}_{11}\right),\ \widehat{\beta} \equiv \log\left(1 - \widehat{h}_{00}\right),$$

for latter convenience. Eq.(32) looks like the metric of a space-time with spherical symmetry in standard coordinates. However there are two important differences. Firstly the metric tensor is a quantum operator rather than a classical (c-number) tensor. Secondly the quantities $\widehat{\alpha}$ and $\widehat{\beta}$ depend on the coordinates θ and ϕ in addition to the dependence on t and r, typical of spherical symmetry.

The quantized metric eq.(32) should be used when working at the atomic scale, but at the cosmic scale we may use a metric obtained by the expectation of the former, that is

$$\begin{aligned} ds^2 &= \langle \Phi \mid d\widehat{s}^2 \mid \Phi \rangle \qquad (33) \\ &= \langle \Phi \mid \exp\left(\widehat{\alpha}\right) \mid \Phi \rangle dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - \langle \Phi \mid \exp(\widehat{\beta}) \mid \Phi \rangle dt^2. \end{aligned}$$

In terms of $\widehat{\alpha}$ and $\widehat{\beta}$ eqs.(30) should be written as follows

$$\frac{8\pi G}{3}\left(\rho_{DE} + \rho_{mat}\right) r^2 \simeq \langle \Phi \mid \exp\left(\widehat{\alpha}\right) \mid \Phi \rangle - 1 \simeq \langle \Phi \mid \widehat{\alpha} + \frac{\widehat{\alpha}^2}{2} \mid \Phi \rangle, \tag{34}$$

$$\frac{8\pi G}{3}\left(\frac{1}{2}\rho_{mat} - \rho_{DE}\right) r^2 \simeq \langle \Phi \mid \exp\left(\widehat{\beta}\right) \mid \Phi \rangle - 1 \simeq \langle \Phi \mid \widehat{\beta} + \frac{\widehat{\beta}^2}{2} \mid \Phi \rangle. \tag{35}$$

Our aim now is to justify these two equations as deriving from vacuum fluctuations. Thus I shall obtain the expectations of the right sides of eqs.(34) and (35) in terms of the correlations (two-point functions) of the density

fluctuations of the vacuum fields. To do that we should begin getting the appropriate quantum Einstein equations (involving tensor operators) and solving them.

From the metric eq.(32) it is straightforward to get the quantum Einstein equations provided we assume that they are similar to the classical counterparts, as explained above. Two of them do not contain time derivatives and they are the only ones to be studied here, that is

$$\begin{aligned} 8\pi G\rho &= G_0^0 = \frac{\alpha}{r^2} - \frac{\alpha^2}{2r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{\alpha}{r}\frac{\partial\alpha}{\partial r} - \frac{1}{2r^2}\cot\theta\frac{\partial\alpha}{\partial\theta} \\ &\quad -\frac{1}{2r^2}\frac{\partial^2\alpha}{\partial\theta^2} - \frac{1}{2r^2s^2}\frac{\partial^2\alpha}{\partial\phi^2} - \frac{1}{4r^2}\left(\frac{\partial\alpha}{\partial\theta}\right)^2 - \frac{1}{4r^2s^2}\left(\frac{\partial\alpha}{\partial\phi}\right)^2, \end{aligned} \tag{36}$$

$$\begin{aligned} -8\pi Gp &= G_1^1 = \frac{\alpha}{r^2} - \frac{\alpha^2}{2r^2} - \frac{1}{r}\frac{\partial\beta}{\partial r} + \frac{\alpha}{r}\frac{\partial\beta}{\partial r} - \frac{1}{2r^2}\cot\theta\frac{\partial\beta}{\partial\theta} \\ &\quad -\frac{1}{2r^2}\frac{\partial^2\beta}{\partial\theta^2} - \frac{1}{2r^2s^2}\frac{\partial^2\beta}{\partial\phi^2} - \frac{1}{4r^2}\left(\frac{\partial\beta}{\partial\theta}\right)^2 - \frac{1}{4r^2s^2}\left(\frac{\partial\beta}{\partial\phi}\right)^2. \end{aligned} \tag{37}$$

Here I have removed the carets of the operators for notational simplicity which will be also made from now on. But I remember that both α, β and their derivatives are quantum operators and that whenever we have a product of two of them it is understood that it means symmetrically orderer product. For instance

$$\frac{\alpha}{r}\frac{\partial\beta}{\partial r} \text{ actually means } \frac{1}{2}\left(\frac{\widehat{\alpha}}{r}\frac{\partial\widehat{\beta}}{\partial r} + \frac{\partial\widehat{\beta}}{\partial r}\frac{\widehat{\alpha}}{r}\right).$$

After some algebra eqs.(36) and (37) may be rewritten, in more compact form,

$$8\pi r^2 G\rho = \alpha - \frac{\alpha^2}{2} + r\frac{\partial\alpha}{\partial r} - r\alpha\frac{\partial\alpha}{\partial r} - \frac{1}{2}\Delta\alpha + \frac{1}{4}\alpha\Delta\alpha - \frac{1}{8}\Delta(\alpha^2), \tag{38}$$

$$-8\pi r^2 Gp = \alpha - \frac{\alpha^2}{2} - r\frac{\partial\beta}{\partial r} + r\alpha\frac{\partial\beta}{\partial r} - \frac{1}{2}\Delta\beta + \frac{1}{4}\beta\Delta\beta - \frac{1}{8}\Delta\left(\beta^2\right), \tag{39}$$

where Δ is the angular part of the Laplacian operator, that is

$$\Delta \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

I shall start solving the nonlinear partial differential eq.(38) , which contains a single unknown, namely the operator $\alpha(r, \theta, \phi)$. Actually α may also depend on time, but as eqs.(36) and (37) do not contain time derivatives the time t appears as a parameter, rather than one of the variables of the partial differential equations, and I shall not write it explicitly. In order to solve eq.(38) I will approximate the solution by a perturbation expansion in powers of the Newton constant G, and work to order $O(G^2)$, writing the (operator) metric parameter α in the form

$$\alpha = G\alpha_0 + G^2\alpha_1, \tag{40}$$

If I put this in eq.(38) the terms of first order in G give the linear (in the unknown α_0) equation

$$8\pi r^2 \rho = \alpha_0 + r\frac{\partial \alpha_0}{\partial r} - \frac{1}{2}\Delta\alpha_0, \tag{41}$$

whilst the terms of second order give an equation also linear (in α_1), namely

$$\alpha_1 + r\frac{\partial \alpha_1}{\partial r} - \frac{1}{2}\Delta\alpha_1 = \frac{\alpha_0^2}{2} + r\alpha_0\frac{\partial \alpha_0}{\partial r} - \frac{1}{4}\alpha_0\Delta\alpha_0 + \frac{1}{8}\Delta\left(\alpha_0^2\right). \tag{42}$$

The solution of eq.(41) , with the condition that $\alpha_0 = 0$ at $r = 0$, may be written in simplified notation as

$$\alpha_0 = 8\pi A(r^2\rho), \tag{43}$$

with the meaning

$$\alpha_0(\mathbf{r}) = 8\pi \int_0^r d^3\mathbf{r}_1 A(\mathbf{r}, \mathbf{r}_1)\left[\mathbf{r}_1^2 \rho(\mathbf{r}_1)\right],\ \mathbf{r} \equiv \{r, \theta, \phi\}, \tag{44}$$

where A is a *kernel* to be specified later on. Eq.(42) , with the condition that $\alpha_1 = 0$ at $r = 0$, may be solved similarly leading to

$$\alpha_1 = A\left[\frac{\alpha_0^2}{2} + r\alpha_0\frac{\partial \alpha_0}{\partial r} - \frac{1}{4}\alpha_0\Delta\alpha_0 + \frac{1}{8}\Delta\left(\alpha_0^2\right)\right], \tag{45}$$

where α_0 is given by eq.(43) . Thus the solution of eq.(38) may be written, in simplified notation (I shall use units $G = 1$ from now on, although writing explicitly Newton´s constant sometimes for the sake of clarity),

$$\alpha = 8\pi A(r^2\rho) + 64\pi^2 A\left[\frac{1}{2}\left(1 + r\frac{\partial}{\partial r} + \frac{1}{4}\Delta\right)[A(r^2\rho)]^2 - \frac{1}{4}[A(r^2\rho)]\Delta[A(r^2\rho)]\right]. \tag{46}$$

We are interested in the expectations eqs.(34) and we get

$$\langle \Phi \mid \alpha + \frac{\alpha^2}{2} \mid \Phi \rangle = \langle \Phi \mid \alpha + \frac{\alpha^2}{2} \mid \Phi \rangle_{mat} + \langle \Phi \mid \alpha + \frac{\alpha^2}{2} \mid \Phi \rangle_{vac}, \tag{47}$$

$$\begin{aligned} \langle \Phi \quad \mid \quad \alpha + \frac{\alpha^2}{2} \mid \Phi \rangle_{mat} &\equiv 8\pi A(r^2 \rho_{mat}) + 32\pi^2 A \left[\left(1 + r\frac{\partial}{\partial r} + \frac{1}{4}\Delta \right) [A(r^2 \rho_{mat})]^2 \right] \\ &-16\pi^2 A \left[[A(r^2\rho_{mat})] \Delta [A(r^2 \rho_{mat})] \right] + 32\pi^2 [A(r^2 \rho_{mat})]^2, \end{aligned} \tag{48}$$

$$\begin{aligned} \langle \Phi \quad \mid \quad \alpha + \frac{\alpha^2}{2} \mid \Phi \rangle_{vac} \equiv 32\pi^2 A \left(1 + r\frac{\partial}{\partial r} + \frac{1}{4}\Delta \right) \langle \Phi \left| [A(r^2 \rho_{vac})]^2 \right| \Phi \rangle, \\ -16\pi^2 \langle \Phi \quad \mid \quad [A(r^2 \rho_{vac})] \Delta [A(r^2 \rho_{vac})] \mid \Phi \rangle + 32\pi^2 \langle \Phi \left| [A(r^2 \rho_{vac})]^2 \right| \Phi \rangle, \end{aligned} \tag{49}$$

where I have taken into account eqs.(24) and (25). The proof is not difficult taking into account that we work to first order in α_1 and to second order in α_0. Let us consider for instance the term

$$\begin{aligned} \langle \Phi \quad &\mid \quad \alpha^2 \mid \Phi \rangle \simeq \langle \Phi \mid \alpha_0^2 \mid \Phi \rangle \\ &= \quad 64\pi^2 \int d^3\mathbf{r}_1 r_1^2 A(\mathbf{r}, \mathbf{r}_1) \int d^3\mathbf{r}_2 r_1^2 A(\mathbf{r}, \mathbf{r}_1) \langle \Phi \left| \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \right| \Phi \rangle. \end{aligned}$$

Taken eqs.(25) and (24) into account the two-point correlation becomes

$$\begin{aligned} \langle \Phi \left| \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \right| \Phi \rangle &= \langle \Phi \left| [\rho_{mat}(\mathbf{r}_1) + \rho_{vac}(\mathbf{r}_1)] [\rho_{mat}(\mathbf{r}_2) + \rho_{vac}(\mathbf{r}_2)] \right| \Phi \rangle \\ &= \rho_{mat}(\mathbf{r}_1) \rho_{mat}(\mathbf{r}_2) + \langle \Phi \left| \rho_{vac}(\mathbf{r}_1) \rho_{vac}(\mathbf{r}_2) \right| \Phi \rangle. \end{aligned}$$

A similar analysis may be made for the other terms. The result is that *the expectation $\langle \Phi \mid \alpha + \alpha^2/2 \mid \Phi \rangle$ is the sum of two expressions, one containing only the matter density, ρ_{mat}, and the other one the vacuum density, ρ_{vac}, i. e. there are no cross terms. It may be also realized that we should solve eq.(38) at least up to terms in G^2 in order to get the leading term due to the vacuum fluctuations.* This is because the expectation of the solution to order G, eq.(44) , gives no contribution due to the vanishing of $\langle vac \left| \widehat{\rho} \right| vac \rangle$, see eq.(24) .

The terms with ρ_{mat} give the contribution of matter density to the metric coefficient α. In particular, if we model the matter density of the universe by

a constant it is not difficult to check, taking eq.(54) into account (see below), that the matter term gives

$$\langle \Phi \mid \alpha + \frac{1}{2}\alpha^2 \mid \Phi \rangle_{mat} \simeq \frac{2GM}{r} + \frac{2G^2M^2}{r^2}, \; M \equiv \frac{4\pi}{3}\rho_{mat} r^3, \tag{50}$$

which agrees with the second order expansion of $\exp \alpha$ in the well known Schwarzschild solution

$$\exp \alpha = \left(1 - \frac{2GM}{r}\right)^{-1}.$$

5 Contribution of the vacuum fluctuations to the metric

In the following I shall calculate the different terms involved in eqs.(49). I start solving eq.(41) by writing $\rho\,(r, \theta, \phi)$ and $\alpha_0\,(r, \theta, \phi)$ as expansions in terms of spherical harmonics, that is

$$\begin{aligned}
\rho &= \sum_{lm} \rho_{lm}\,(r)\, Y_{lm}\,(\theta, \phi) \Rightarrow \; \rho_{lm}\,(r) \equiv \int \rho\,(r, \theta, \phi)\, Y^*_{lm}\,(\theta, \phi)\, d\Omega, \\
\alpha_0 &= \sum_{lm} a_{lm}\,(r)\, Y_{lm}\,(\theta, \phi) \Rightarrow a_{lm}\,(r) \equiv \int \alpha_0\,(r, \theta, \phi)\, Y^*_{lm}\,(\theta, \phi)\, d\Omega. \quad (51)
\end{aligned}$$

I get from eq.(41)

$$8\pi \rho_{lm} r^2 = \left[1 + \frac{1}{2} l\,(l+1)\right] a_{lm} + r\frac{da_{lm}}{dr}, \tag{52}$$

whose solution with the initial condition $a_{lm}\,(0) = 0$ is

$$a_{lm}\,(r) = 8\pi r \int_0^r (y/r)^{2+l(l+1)/2} \rho_{lm}\,(y)\, dy, \tag{53}$$

Taking eqs.(51) into account I get (see eq.(44))

$$A(\mathbf{r}, \mathbf{r_1}) \equiv r^{-1} \sum_{lm} Y_{lm}\,(\theta, \phi)\,(r_1/r)^{l(l+1)/2} Y^*_{lm}\,(\theta_1, \phi_1)\,. \tag{54}$$

I am interested in the expectations involving ρ_{vac} defined in eq.(49) , where I shall write $\langle . \rangle$ for $\langle \Phi \,|.|\, \Phi \rangle$ for notational simplicity. I begin with

$$\begin{aligned} \langle \alpha^2 \rangle &\simeq \langle \alpha_0^2 \rangle = \langle (A[r^2\rho])^2 \rangle = \int_0^r dr_1 \int d\Omega_1 r_1^2 A(\mathbf{r}, \mathbf{r_1}) \\ &\times \int_0^r dr_2 \int d\Omega_2 r_2^2 A(\mathbf{r}, \mathbf{r_2}) C(s), \end{aligned} \tag{55}$$

where the two-point correlation function of the density, $C(s)$, was given in eq.(12) with

$$s \equiv |\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 u},\ u \equiv \cos\theta_{12}. \tag{56}$$

A consequence of the assumption that the correlation C depends only on the distance $|\mathbf{r}_1 - \mathbf{r}_2|$, rather than on $\mathbf{r}_1$ and $\mathbf{r}_2$ separately, is that the expectation eq.(55) will not depend on the angular variables (θ, ϕ) and we may average over these variables. Hence it follows that $l' = l, m' = m$ and we get

$$\begin{aligned} \langle (A[r^2\rho])^2 \rangle &= r^2 \int_0^r dr_1 \int d\Omega_1 \int_0^r dr_2 \int d\Omega_2 \\ & \sum_{lm} (r_1 r_2 / r^2)^{l(l+1)/2+2} Y_{lm}(\theta_1, \phi_1)\, Y_{lm}^*(\theta_2, \phi_2)\, C(s). \end{aligned} \tag{57}$$

This leads to

$$\langle (A[r^2\rho])^2 \rangle = \frac{r^2}{4\pi} \int_0^r dr_1 \int_0^r dr_2 \int_{-1}^1 du \sum_l (2l+1)\, (r_1 r_2/r^2)^{2+l(l+1)/2} P_l(u)\, C(s). \tag{58}$$

where I have taken into account the following property of spherical harmonic functions

$$\sum_m Y_{lm}(\theta_1, \phi_1)\, Y_{lm}^*(\theta_2, \phi_2) = \frac{2l+1}{4\pi} P_l(\cos\theta_{12}), \tag{59}$$

θ_{12} being the angle between the directions (θ_1, ϕ_1) and (θ_2, ϕ_2) and $P_l(u)$ a Legendre polynomial. Now I introduce the new variables $\{y, z\}$ defined by

$$r_1 = y + z/2,\ r_2 = y - z/2, \tag{60}$$

substitute an s integration for the u integration and change the order of the integrals. Thus eq.(58) becomes after some algebra

$$\begin{aligned}\langle\alpha^2\rangle &\simeq \langle (A[r^2\rho])^2\rangle \simeq \frac{1}{4\pi}\int_0^\infty C(s)sds\sum_l (2l+1)\int_0^s dz\ I_l(r,z) \qquad (61)\\ I_l(r,z) &\equiv \int_{z/2}^{r-z/2} dy\left(\frac{y^2-z^2/4}{r^2}\right)^{1+l(l+1)/2} P_l\left(1-\frac{s^2-z^2}{2y^2-z^2/2}\right).\end{aligned}$$

A similar method allows calculating the term $\langle\Phi\,|(A\rho_{vac})\Delta(A\rho_{vac})|\,\Phi\rangle$. Putting all relevant terms of eq.(49) together we obtain

$$\langle\alpha\rangle = 8\pi A\left[\int_0^\infty C(s)sds\sum_l (2l+1)\int_0^s dz\left(1+r\frac{\partial}{\partial r}+\frac{1}{2}l\,(l+1)\right)I_l(r,z)\right]. \qquad (62)$$

It may be realized that the main contribution to $\langle\alpha\rangle$ comes from high values of l. Thus we might neglect $1 \ll \frac{1}{2}l\,(l+1)$, which also shows that the term $\frac{1}{2}\alpha_0^2$ is negligible in comparison with the term $-\frac{1}{4}\alpha_0\Delta\alpha_0$ in eq.(45). Similarly we may neglect

$$\langle\frac{1}{2}\alpha^2\rangle \ll \langle\alpha\rangle \qquad (63)$$

in eq.(49). Proceeding with the calculation of eq.(62), I get after some algebra

$$\begin{aligned}\left(1+r\frac{\partial}{\partial r}+\frac{1}{2}l\,(l+1)\right)I_l(r,z) &= \left(\frac{r-z}{r}\right)^{1+l(l+1)/2} P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\\ &\quad -\left[1+\frac{1}{2}l\,(l+1)\right]I_l(r,z), \qquad (64)\end{aligned}$$

where I have neglected $z^2/2 \ll 2r^2$ in the argument of the Legendre polynomial. In the quantity $I_l(r,z)$ we may approximate $2y^2 - z^2/2$ by $2r^2$ because only values of y close to r contribute substantially to the y-integral. This

leads to

$$\begin{aligned}
I_l &\simeq P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\int_{z/2}^{r-z/2} dy(y/r)^{2+l(l+1)}\exp\left(-\frac{z^2}{8y^2}\left[2+l(l+1)\right]\right)\\
&\simeq P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\exp\left(-\frac{z^2}{8r^2}\left[2+l(l+1)\right]\right)\int_{z/2}^{r-z/2} dy(y/r)^{2+l(l+1)}\\
&\simeq r\left[3+l\left(l+1\right)\right]^{-1}P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\\
&\quad\times\exp\left(-\frac{z^2}{8r^2}\left[2+l(l+1)\right]\right)\left(\frac{r-z/2}{r}\right)^{2+l(l+1)}\\
&\simeq r\left[3+l\left(l+1\right)\right]^{-1}P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\exp\left(-\frac{z}{2r}\left[2+l(l+1)\right]\right), \qquad (65)
\end{aligned}$$

where I have neglected

$$\frac{z^2}{8r^2}\left[2+l(l+1)\right]\ll\frac{z}{2r}\left[2+l(l+1)\right]$$

in the exponential because $z<s\ll r$. Hence eq.(64) becomes

$$\begin{aligned}
\langle\alpha\rangle &\simeq 4\pi A\left[r\int_0^\infty C(s)sds\sum_l\left(2l+1\right)\frac{4+l(l+1)}{3+l(l+1)}\right.\\
&\quad\left.\times\int_0^s dz\exp\left(-\frac{z}{2r}\left[2+l(l+1)\right]\right)P_l\left(1-\frac{s^2-z^2}{2r^2}\right)\right]. \qquad (66)
\end{aligned}$$

We may substitute a Bessel function for the Legendre polynomial, which is a good approximation for large l and argument close to unity, as is shown by the well known limit (see Gradshteyn et al.[13], n^o 8.722-2)

$$\lim_{l\to\infty}P_l\left(\cos\frac{x}{l}\right)=J_0\left(x\right).$$

After that I will substitute an integral for the sum in l and I get

$$\begin{aligned}
\langle\alpha\rangle &\simeq 4\pi A\left[r\int_0^\infty C(s)sds\int_0^s dz\int_0^\infty ldl\exp\left[-zl^2/2r\right]J_0\left(\frac{l\sqrt{s^2-z^2}}{r}\right)\right]\\
&= 4\pi A\left[r^2\int_0^\infty C(s)sds\int_0^s\frac{dz}{z}\exp\left[\frac{z^2-s^2}{2zr}\right]\right], \qquad (67)
\end{aligned}$$

where the l integral has been taken from the literature (see Gradshteyn[13] n^o 6.631-4) . The z integral may be performed in terms of the new variable $t = s^2/(2rz)$, which leads to

$$\begin{aligned}\int_0^s \frac{dz}{z} \exp\left[\frac{z^2 - s^2}{2rz}\right] &= \int_{s/2r}^{\infty} \frac{dt}{t} \exp\left[\frac{s^2}{2r^2 t} - t\right] = \int_{s/2r}^{\infty} \frac{dt}{t} \exp\varepsilon \exp\left[-t\right] \\ &\simeq \int_{s/2r}^{\infty} \frac{dt}{t} \exp\left[-t\right] = \log(2r/s) - 0.557, \end{aligned} \tag{68}$$

where I have put $\exp\varepsilon \simeq 1$ because $\varepsilon \equiv s^2/\left(2r^2 t\right) \ll 1$ for $t \geq s/2r$. Now, taking into account the action of the kernel A, eq.(54) , applied to a function $f\left(\mathbf{r}\right)$ not depending on the polar angles $\{\theta, \phi\}$, we get finally

$$\begin{aligned}\langle\Phi \quad | \quad \widehat{g}_{11} \mid \Phi\rangle_{vac} &\simeq \langle\Phi \mid \alpha + \frac{\alpha^2}{2} \mid \Phi\rangle_{vac} \simeq \langle\Phi \mid \alpha \mid \Phi\rangle_{vac} \qquad (69) \\ &\simeq 2\pi r^2 \int_0^{\infty} C(s) s ds \left(\log(2r/s) - 0.557\right) \sim 600 r^2 \int_0^{\infty} C(s) s ds,\end{aligned}$$

where I have taken into account eq.(63) and I have estimated $r/s \sim 10^{40}$. If we include the contribution of matter and put explicitly Newton´s constant we get

$$\langle\Phi \mid \alpha + \frac{\alpha^2}{2} \mid \Phi\rangle \simeq \frac{8\pi G}{3} \rho_{mat} r^2 + 600 G^2 r^2 \int_0^{\infty} C(s) s ds. \tag{70}$$

Now I shall solve the second component of the Einstein equation, eq.(39) , which involves both coefficients α and β of the metric. I will search for a solution of the form

$$\beta(r, \theta, \phi) = -\alpha(r, \theta, \phi) + \gamma\left(\theta, \phi\right) r^2 + O\left(r^4\right),$$

which taking eq.(38) into account leads to

$$\begin{aligned}8\pi r^2 \left(\rho + p\right) &= 2r^2\gamma - 2r^2\alpha\gamma + \frac{1}{2} r^2 \Delta\gamma - \Delta\alpha + \frac{1}{4}\alpha\Delta\alpha \\ &\quad -\frac{1}{4}(\alpha - \gamma r^2)\Delta(\alpha - \gamma r^2) - \frac{1}{8}\Delta(\alpha^2) + \frac{1}{8}\Delta[(\alpha - \gamma r^2)^2] \\ &= 2r^2\gamma - \Delta\alpha + O\left(r^4\right).\end{aligned}$$

where in the second equality I have taken into account that α is of order $O\left(r^2\right)$ (see eqs.(51) and (53)). This leads to the solution

$$\beta = -\alpha + 4\pi r^2 \left(\rho + p\right) + \frac{1}{2}\Delta\alpha + O\left(r^4\right).$$

Hence eqs.(34) and (35) lead to the following equalities, where terms of order $O(r^4)$ are neglected

$$\begin{aligned} -\langle\Phi\left|\widehat{\alpha}+\frac{\widehat{\alpha}^2}{2}\right|\Phi\rangle &\simeq -\langle\Phi\mid\widehat{\alpha}\mid\Phi\rangle\simeq\langle\Phi\mid\widehat{\beta}\mid\Phi\rangle-4\pi r^2\rho_{mat} \\ &\simeq \langle\Phi\mid\widehat{\beta}+\frac{\widehat{\beta}^2}{2}\mid\Phi\rangle-4\pi r^2\rho_{mat}. \end{aligned}$$

The third equality takes into account that $\langle\Phi\mid\widehat{\alpha}\mid\Phi\rangle$ does not depend on the angles θ,ϕ, so that $\langle\Phi\mid\Delta\widehat{\alpha}\mid\Phi\rangle=\Delta\langle\Phi\mid\widehat{\alpha}\mid\Phi\rangle=0$, and the last equality the fact that β is of order $O(r^2)$. Hence, taking eq.(70) into account, we get

$$\langle\Phi\mid\beta+\frac{\beta^2}{2}\mid\Phi\rangle\simeq\frac{4\pi G}{3}\rho_{mat}r^2-600G^2r^2\int_0^\infty C(s)sds. \tag{71}$$

6 Conclusions

The main result of the paper is that eqs.(70) and (71) provide the expectation of the quantized metric (see eq.(33)), in terms of the two-point correlation of the vacuum fluctuations, $C(s)$, that is

$$\begin{aligned} ds^2 &= \left(\frac{8\pi G}{3}\rho_{mat}r^2+600G^2r^2\int_0^\infty C(s)sds\right)dr^2+r^2d\theta^2 \\ &\quad +r^2\sin^2\theta d\phi^2-\left(\frac{4\pi G}{3}\rho_{mat}r^2-600G^2r^2\int_0^\infty C(s)sds\right)dt^2. \end{aligned} \tag{72}$$

This is identical to the standard FRW metric, eq.(1), via the approximate "free falling" metric eq.(6), provided that we identify

$$\rho_{DE}\simeq 140G\int_0^\infty C(s)sds, \tag{73}$$

which shows that vacuum fluctuations give rise to a curvature of space-time similar to what would be produced by a "dark energy" density (plus cold matter). However we cannot fix the value of ρ_{DE} as far as we do not know the two-point correlation function of vacuum density fluctuations. Crucial for the result is the fact that Einstein´s equation involves a *non-linear* relation between the stress-energy tensor and the metric tensor. In fact, if the relation was linear, then the vanishing of the vacuum expectation of the

quantum matter density operator would imply vanishing of curvature, that is Minkowski space, in the absence of matter.

Our results suggest that the observed accelerated expansion of the universe might be explained as due to the quantum vacuum fluctuations, without the need of any "dark energy". If this is the case dark energy, ρ_{DE}, appears as a parameter which mimics the effect of the vacuum fluctuations. The derived relation between the two-point correlation of the fluctuations and the value of the dark energy parameter would allow calculating the latter if the said two-point correlation was known. Arguments are given which suggest that such calculation might give results in agreement with observations.

7 Appendix. Two-point correlation of vacuum fluctations of the radiation field.

Let us start calculating the two-point correlation

$$A(\mathbf{r}_1,t_{1;}\mathbf{r}_2,t_2) \equiv \left\langle vac \left| (:\widehat{\mathbf{E}}\left(\mathbf{r}_1,t_1\right)^2:)(:\widehat{\mathbf{E}}\left(\mathbf{r}_2,t_2\right)^2:) \right| vac \right\rangle . \tag{74}$$

Taking eqs.(17) into account it is not difficult to prove that contributions to A, eq.(74) , will derive only from terms with two annihilation operators coming from $(:\widehat{\mathbf{E}}\left(\mathbf{r}_1,t_1\right)^2:)$ and two creation operators coming from $(:\widehat{\mathbf{E}}\left(\mathbf{r}_2,t_2\right)^2:)$. Thus the contributing terms will derive from

$$\begin{aligned}
&\frac{\hbar}{V}\sum_{\mathbf{k}\boldsymbol{\varepsilon}}\sum_{\mathbf{k}'\boldsymbol{\varepsilon}'}\sqrt{\omega\omega'}a_{\mathbf{k}\boldsymbol{\varepsilon}}a_{\mathbf{k}'\boldsymbol{\varepsilon}'}\boldsymbol{\varepsilon}\left(\mathbf{k}\right)\cdot\boldsymbol{\varepsilon}'\left(\mathbf{k}'\right)\exp\left[i\left(\mathbf{k}'+\mathbf{k}\right).\mathbf{r}_1-i\omega t_1\right]\\
&\times\frac{\hbar}{V}\sum_{\mathbf{k}''\boldsymbol{\varepsilon}''}\sum_{\mathbf{k}'''\boldsymbol{\varepsilon}'''}\sqrt{\omega''\omega'''}a^{+}_{\mathbf{k}''\boldsymbol{\varepsilon}''}a^{+}_{\mathbf{k}'''\boldsymbol{\varepsilon}'''}\boldsymbol{\varepsilon}''\left(\mathbf{k}''\right)\cdot\boldsymbol{\varepsilon}'''\left(\mathbf{k}'''\right)\\
&\times\exp\left[-i\left(\mathbf{k}''+\mathbf{k}'''\right).\mathbf{r}_2+i\omega t_2\right].
\end{aligned}$$

Taking into account that

$$\left\langle vac\left|a_{\mathbf{k}\boldsymbol{\varepsilon}}a_{\mathbf{k}'\boldsymbol{\varepsilon}'}a^{+}_{\mathbf{k}''\boldsymbol{\varepsilon}''}a^{+}_{\mathbf{k}'''\boldsymbol{\varepsilon}'''}\right|vac\right\rangle=\delta_{\mathbf{k}\mathbf{k}''}\delta_{\mathbf{k}'\mathbf{k}'''}+\delta_{\mathbf{k}\mathbf{k}'''}\delta_{\mathbf{k}'\mathbf{k}''},$$

we get

$$\begin{aligned}
A &= \frac{2\hbar^2}{V^2}\sum_{\mathbf{k}\boldsymbol{\varepsilon}}\sum_{\mathbf{k}'\boldsymbol{\varepsilon}'}\omega\omega'\left[\boldsymbol{\varepsilon}\left(\mathbf{k}\right)\cdot\boldsymbol{\varepsilon}'\left(\mathbf{k}'\right)\right]^2\exp\left[i\left(\mathbf{k}'+\mathbf{k}\right).\mathbf{r}-i\left(\omega+\omega'\right)t\right]\\
&= \frac{2\hbar^2}{V^2}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\omega\omega'\left(1+\frac{\left(\mathbf{k}\cdot\mathbf{k}'\right)^2}{\mathbf{k}^2\mathbf{k}'^2}\right)\exp\left[i\left(\mathbf{k}'+\mathbf{k}\right).\mathbf{r}-i\left(\omega+\omega'\right)t\right]
\end{aligned} \tag{75}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, t \equiv t_1 - t_2$ and I have performed the sum in polarizations in the second equality. I shall use polar angles taking the direction of the vector $\mathbf{r}$ as polar axis, so that

$$\mathbf{k} \equiv \omega \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right), \mathbf{k} \equiv \omega' \left(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'\right).$$

After the standard replacements

$$\frac{1}{V}\sum_{\mathbf{k}} \rightarrow \frac{1}{8\pi^3}\int_0^\infty \omega^2 d\omega \int_{-1}^{1} d\left(\cos\theta\right)\int_0^{2\pi} d\phi,$$

eq.(75) gives

$$\begin{aligned} A &= \frac{\hbar^2}{32\pi^6}\int_0^\infty \omega^3 d\omega \int_0^\infty \omega'^3 d\omega' \int_{-1}^{1} d\left(\cos\theta'\right)\int_0^{2\pi} d\phi' \int_{-1}^{1} d\left(\cos\theta\right)\int_0^{2\pi} d\phi \\ &\quad \times \exp\left[i\omega'\left(r\cos\theta' - t\right)\right]\exp\left[i\omega'\left(r\cos\theta' - t\right)\right] \\ &\quad \times \left[1 + \left(\sin\theta\cos\phi\sin\theta'\cos\phi' + \sin\theta\sin\phi\sin\theta'\sin\phi' + \cos\theta\cos\theta'\right)^2\right]. \end{aligned}$$

Integrating the angles ϕ and ϕ' we get

$$\begin{aligned} A &= \frac{\hbar^2}{16\pi^4}\int_0^\infty \omega^3 d\omega \int_0^\infty \omega'^3 d\omega' \int_{-1}^{1} du' \int_{-1}^{1} du \left[3 + 3u^2u'^2 - u^2 - u'^2\right] \\ &\quad \times \exp\left[i\omega\left(ru' - t\right)\right]\exp\left[i\omega'\left(ru - t\right)\right], \end{aligned}$$

where $r \equiv |\mathbf{r}|$, $u \equiv \cos\theta, u' \equiv \cos\theta'$. The u and u' integrals are trivial and I obtain

$$\begin{aligned} A &= \frac{\hbar^2}{16\pi^4}\int_0^\infty \omega^3 d\omega \int_0^\infty \omega'^3 d\omega' \exp\left[-i\left(\omega + \omega'\right)t\right] \\ &\quad \times \left[3I\left(x\right)I\left(x'\right) + 3J\left(x\right)J\left(x'\right) - I\left(x\right)J\left(x'\right) - J\left(x\right)I\left(x'\right)\right], \quad (76) \end{aligned}$$

where

$$\begin{aligned} I\left(x\right) &\equiv \int_{-1}^{1} du \exp\left(-ixu\right) = \frac{2\sin x}{x}, \; x \equiv \omega r, \\ J\left(x\right) &\equiv -\frac{d^2}{dx^2}I\left(x\right) = -\frac{2\sin x}{x} - \frac{4\cos x}{x^2} + \frac{4\sin x}{x^3}. \end{aligned}$$

Now

$$\begin{aligned}\int_0^\infty \omega^3 d\omega \exp(-i\omega t) I(x) &= \frac{2}{r^4}\int_0^\infty x^2 d\omega \exp(-ixt/r)\sin x = \frac{2(3t^2+r^2)}{(r^2-t^2)^3},\\ \int_0^\infty \omega^3 d\omega \exp(-i\omega t) J(x) &= -\int_0^\infty \omega^3 d\omega \exp(-ixt/r)\frac{d^2}{dx^2}\left(\frac{2\sin x}{x}\right)\\ &= \frac{2(t^2+3r^2)}{(r^2-t^2)^3}.\end{aligned}$$

The correlation between the magnetic energies is the same as eq.(74) .The correlation between electric and magnetic parts may be derived without difficulty. It is

$$\begin{aligned}B &= \frac{2\hbar^2}{V^2}\sum_{\mathbf{k}\varepsilon}\sum_{\mathbf{k}'\varepsilon'}\omega\omega'\left[i\left(\mathbf{k_r}\times\boldsymbol{\varepsilon}(\mathbf{k})\right)\cdot\boldsymbol{\varepsilon}'(\mathbf{k}')\right]^2\exp\left[i(\mathbf{k}'+\mathbf{k}).\mathbf{r}-i(\omega+\omega')t\right],\\ \mathbf{k_r} &= \frac{\mathbf{k}.\mathbf{r}}{\omega r^2}\mathbf{r}\equiv\omega(0,0,\cos\theta),\end{aligned}$$

whence

$$\begin{aligned}B &= -\frac{2\hbar^2}{V^2}\sum_{\mathbf{k}\varepsilon}\sum_{\mathbf{k}'\varepsilon'}\omega\omega'\left(1+\cos^2\theta\right)\exp\left[i(\mathbf{k}'+\mathbf{k}).\mathbf{r}-i(\omega+\omega')t\right]\\ &\to -\frac{\hbar^2}{8\pi^4}\int_0^\infty\omega^3 d\omega\int_0^\infty\omega'^3 d\omega' \qquad (77)\\ &\quad\times\left[I(x)I(x')+J(x)J(x')\right]\exp\left[-i(\omega+\omega')t\right]. \qquad (78)\end{aligned}$$

Taking into account eqs.(76) to (79) we finally obtain

$$C(r,t) = 2A+B = \frac{2\hbar^2}{\pi^4(r^2-t^2)^4}. \qquad (79)$$

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